Information-Theoretic Benchmarks for Coded Modulation

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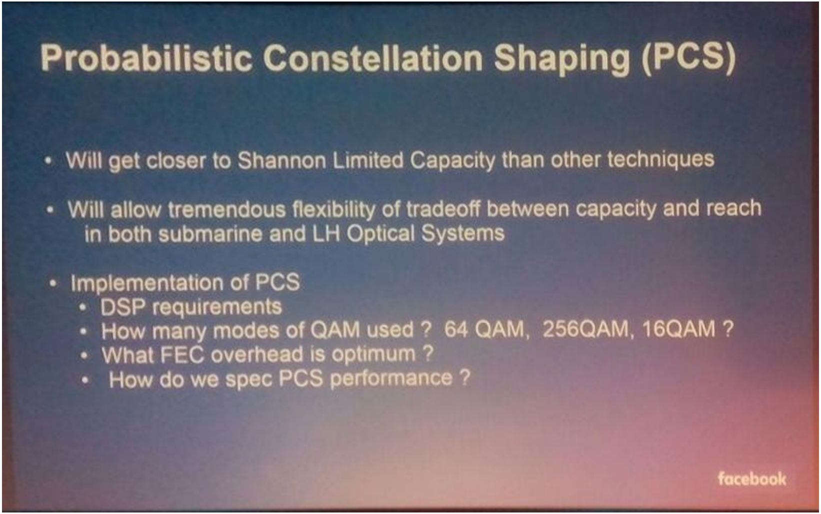
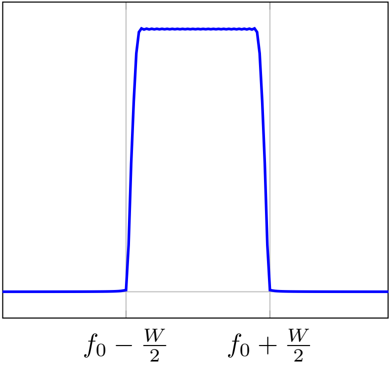
Steve Grubb on Probabilistic Shaping

Global Optical Architecture at **Facebook**, current work focussed on building new worldwide submarine fiber optics networks.



Outline

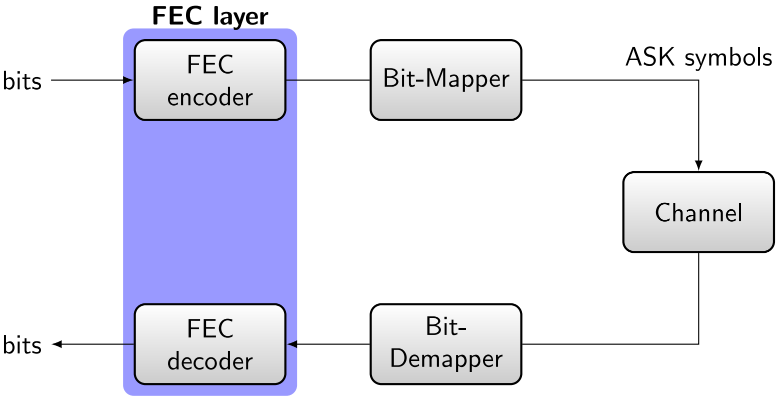
Higher-Order Modulation



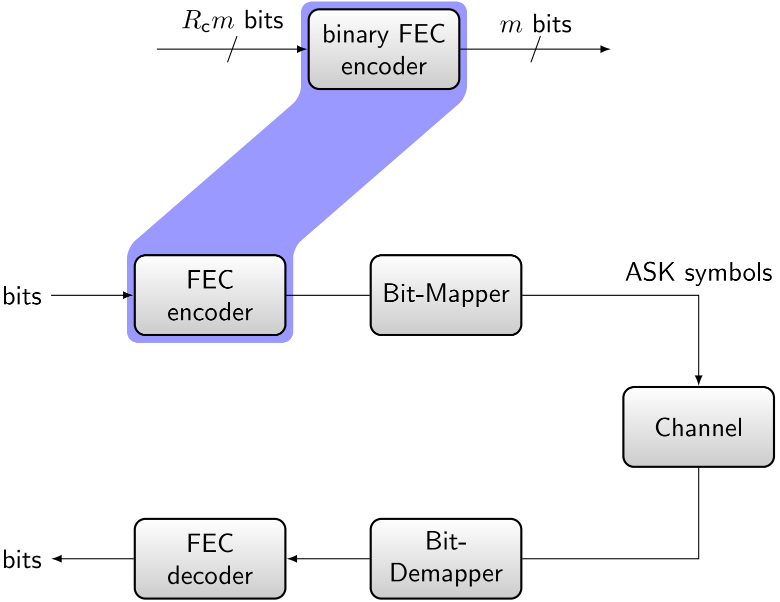
Grid width $W = 50$ GHz, dual polarization.

	QPSK		higher-order modulation			
spectral efficiency $\left[\frac{\text{bits}}{\text{QAM symbol}}\right]$	1	2	3	4	5	6
max net data rate $\left[\frac{\text{Gbit}}{\text{s}}\right]$	100	200	300	400	500	600

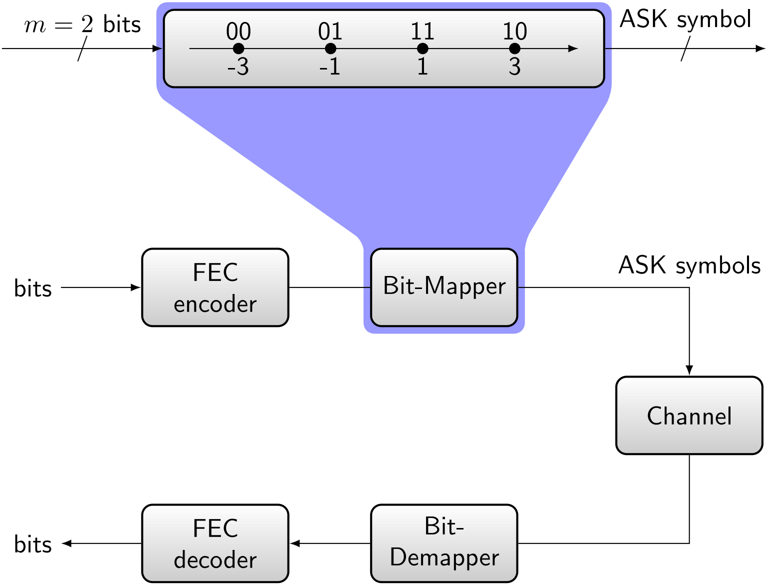
Coded Modulation



Coded Modulation: Binary FEC Code

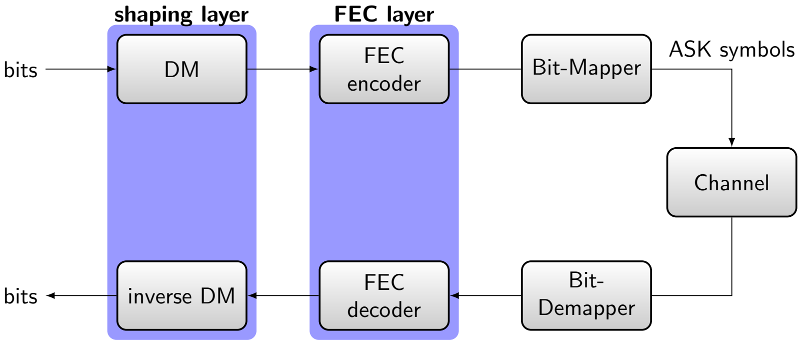


Coded Modulation: Example Bitmapper

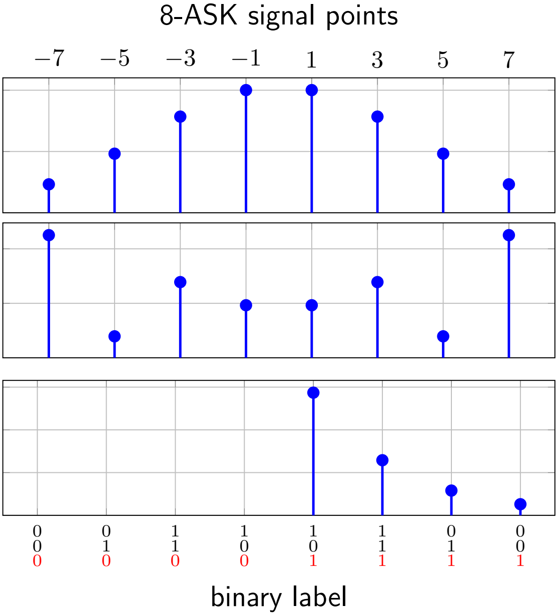


Outline

Layered Probabilistic Shaping



Good Input Distributions



Enabling Component: Systematic Encoder

- Every linear code[1] has a systematic encoder of the form:
  - Copy input unchanged to output.
  - Append parity bits.
- In matrix representation:

$$\boldsymbol{y} = \boldsymbol{u}\boldsymbol{G} = \boldsymbol{u}[\boldsymbol{I}|\boldsymbol{P}] = \boldsymbol{u}\boldsymbol{s}$$

  - Generator matrix $\boldsymbol{G}$, identity matrix $\boldsymbol{I}$, parity forming part $\boldsymbol{P}$.
  - Source bits $\boldsymbol{u}$.
  - Parity bits $\boldsymbol{s} = \boldsymbol{u}\boldsymbol{P}$.

[1] Up to column permutations

Binary FEC Encoder for Probabilistic Amplitude Shaping

Systematically encode the $m-1$ amplitude bits:

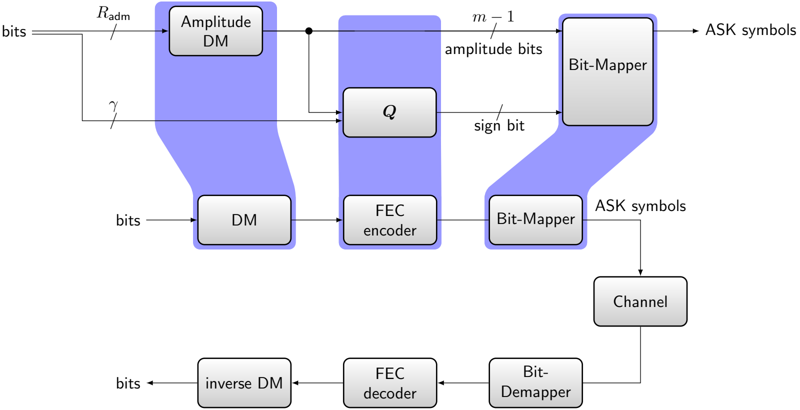
- Binary $(R_{\text{bc}}mn, mn)$ code.
- Dimension $R_{\text{bc}}mn \geq (m-1)n$

$$\Rightarrow R_{\text{bc}} = 1 - \frac{1}{m} + \frac{\gamma}{m}, \quad 0 \leq \gamma \leq 1.$$

- $(m-1)n$ systematic bits $\Rightarrow$ encode with generator matrix

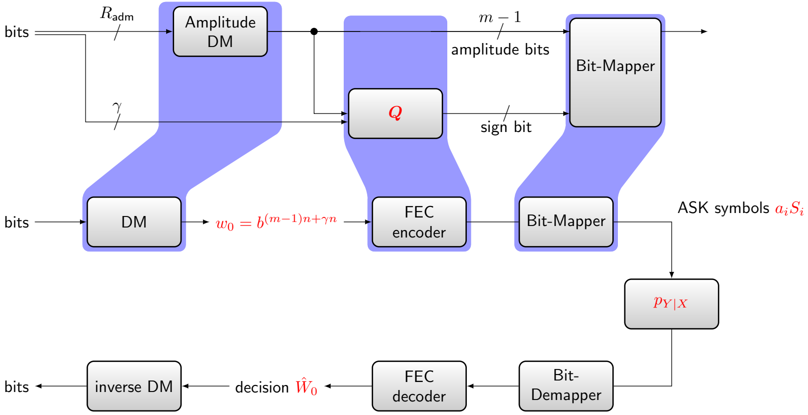
$$\boldsymbol{G} = [\boldsymbol{I}|\boldsymbol{Q}], \quad \boldsymbol{I} \text{ is } R_{\text{bc}}mn \times (m-1)n \text{ upper identity matrix.}$$

Probabilistic Amplitude Shaping



Outline

Random PAS Ensemble



FEC Decoder

- Demapper calculates non-negative metric

$$q(x, y), \quad x \in \mathcal{X}, y \in \mathcal{Y}.$$

- Decoder calculates

$$\hat{W} = \underset{w \in \{0,1\}^{R_{\text{bc}} mn}}{\operatorname{argmax}} \prod_{i=1}^{n} q(X_i(w), y_i)$$

- Decoding error probability

$$P_e = \Pr(\hat{W} \neq w_0).$$

- How large can we make the code rate $R_{\text{c}} = R_{\text{bc}} m$ while ensuring $P_e = 0$ for large n?

Achievable Code Rate

- **Achievable code rate**:

$$R_{\text{ac}} = \underbrace{\log_2 |\mathcal{X}|}_{=m} - \underbrace{\mathbb{E}\left[-\log_2 \frac{q(X,Y)}{\sum_{x\in\mathcal{X}} q(x,Y)}\right]}_{\text{uncertainty}}$$

- $P_X = P_A P_U$
  - P_A = empirical amplitude distribution of a^n.
  - P_U = uniform sign distribution on $\{-1, 1\}$.

Achievable Binary Code (ABC) Rate

- Represent input X by binary label $\boldsymbol{B} = B_1 B_2 \ldots B_m = \text{label}(X)$.
- **ABC rate:**

$$R_{\text{abc}} = 1 - \underbrace{\frac{1}{m} \sum_{i=1}^{m} \mathbb{E}\left[-\log_2 \frac{q_i(B_i, Y)}{\sum_{b \in \{0,1\}} q_i(b, Y)} \right]}_{\text{binary uncertainty (buc)}}$$

Achievable Code Rate Instances

symbol-metric	achievable code rate R_{ac}
optimal	$m - \mathbb{H}(X\|Y)$
bit-metric	**ABC rate** R_{abc}
optimal	$1 - \frac{1}{m}\sum_{j=1}^{m}\mathbb{H}(B_j\|Y)$
bit-interleaved	$1 - \mathbb{H}(B\|Y)$
Hamming metric	$1 - \mathbb{H}(\epsilon)$
⋮	

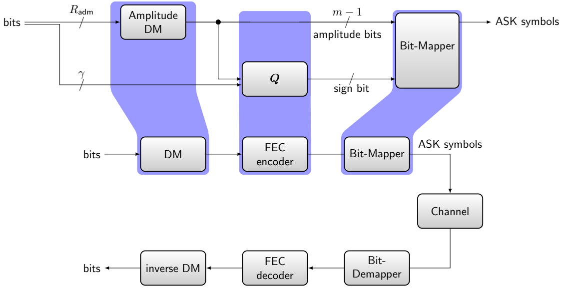
- All follows by $\mathbb{D}(P\|P') \geq 0$ with equality iff $P = P'$.

Outline

- Achievable code rate holds for every amplitude sequence a^n with empirical distribution P_A.

$\Rightarrow$ Encode into permutations of a^n.

Distribution Matching (DM) Rate



- ADM maps k_{adm} input bits to n_{adm} amplitudes:

$$d_1 d_2 \dots d_{k_{\text{adm}}} \text{ bits} \rightarrow \boxed{\text{ADM}} \rightarrow a_1 a_2 \dots a_{n_{\text{adm}}} \text{ amplitudes}$$

- Empirical output distribution P_A.
- Rate $R_{\text{adm}} = \frac{k_{\text{adm}}}{n_{\text{adm}}}$.

PAS Rate

- Code rate $R_{\mathrm{c}} = mR_{\mathrm{bc}} = m - 1 + \gamma$.
- Achievable code rate

$$R_{\mathrm{ac}} = m - \text{uncertainty}.$$

- Rate $R = R_{\mathrm{adm}} + \gamma$.
- Achievable rate

$$\boxed{R_{\mathrm{pas}} = [R_{\mathrm{adm}} + 1 - \text{uncertainty}]^{+}}.$$

CCDM Rate Loss

- Constant Composition Distribution Matcher (CCDM) indexes all length n sequences of distribution P_A.
- CCDM Rate loss

$$\mathbb{H}(P_A) - R_{\text{ccdm}}(P_A, n) \in \Theta\left(\frac{\log n}{n}\right).$$

$\Rightarrow$ For large n, PAS rate is

$$\boxed{R_{\text{pas}} = [\mathbb{H}(P_X) - \text{uncertainty}]^+}.$$

where $\mathbb{H}(P_X) = \mathbb{H}(P_A) + 1$.

PAS Rate Instances

symbol-metric	PAS Rate R_{pas}
optimal	$[\mathbb{H}(X) - \mathbb{H}(X\|Y)]^+ = \mathbb{I}(X;Y)$
bit-metric	
optimal	$\left[\mathbb{H}(\boldsymbol{B}) - \sum_{j=1}^{m} \mathbb{H}(B_j\|Y)\right]^+$
bit-interleaved	$[\mathbb{H}(\boldsymbol{B}) - m\,\mathbb{H}(B\|Y)]^+$
Hamming metric	$[\mathbb{H}(\boldsymbol{B}) - m\,\mathbb{H}(\epsilon)]^+$
⋮	

- **Information-theoretic remark:**
  By $\mathbb{I}(X;Y)$, PAS can achieve capacity with
  1. non-uniform P_X,
  2. linear code,
  3. no alphabet extension.

  Gallager (1968) can do 1, 2 or 1, 3 or 2, 3, but not 1, 2, 3.

Outline

Parametrization of AWGN Channel

- 8-ASK constellation:

$$\mathcal{X} = \{\pm 1, \pm 3, \pm 5, \pm 7\}.$$

- Input X with distribution P_X on $\mathcal{X}$.
- Binary label

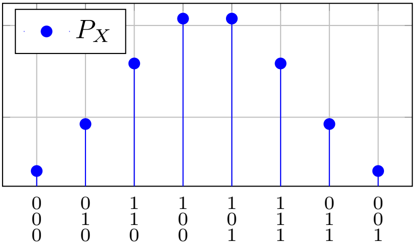
$$\boldsymbol{B} = B_1 B_2 B_3 = \text{label}(X).$$

- Channel output

$$Y = \Delta X + Z.$$

- Noise Z with variance σ^2.
- Signal-to-noise ratio (SNR)

$$\text{SNR} = \frac{\mathbb{E}\left[(\Delta X)^2\right]}{\sigma^2}.$$



Fixed Overhead

- Binary code rate $R_{\text{bc}} = 5/6$ (20% FEC overhead)
- For each SNR, choose P_X, Δ:
  - Maximize $\mathbb{H}(A)$.
  - **Subject to**
    1. SNR constraint:

$$\frac{\mathbb{E}[(\Delta X)^2]}{\sigma^2} = \text{SNR}$$

    2. Uncertainty constraint:

$$\frac{1}{m}\sum_{j=1}^{m} \mathbb{H}(B_j|\Delta X + Z) = 1 - R_{\text{bc}}.$$

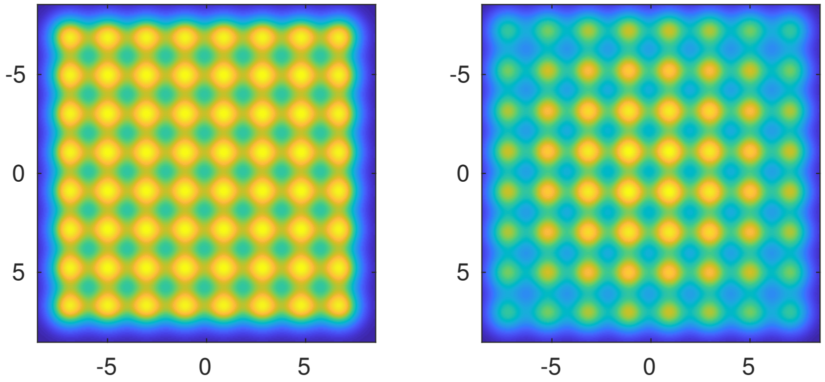
- Uncertainty is the same for all SNRs $\Rightarrow$ **fixed FEC overhead**.

Scatterplot of Received 64-QAM

On the next slides:

- Scatterplot as heatmap of 64-QAM constellation superposed with AWGN.

16,0 dB



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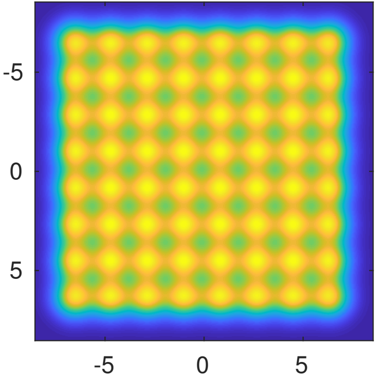

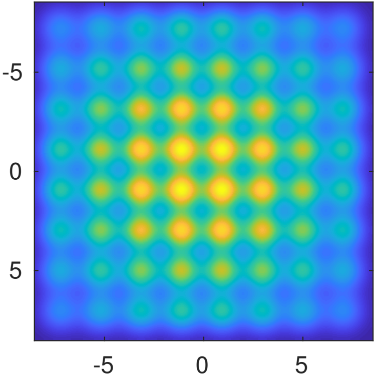

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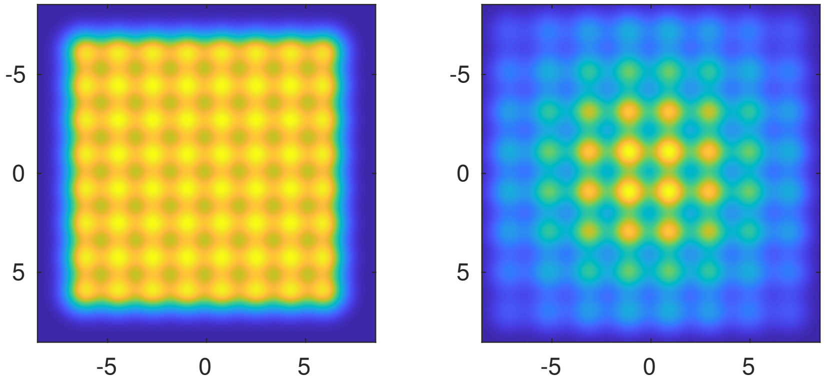

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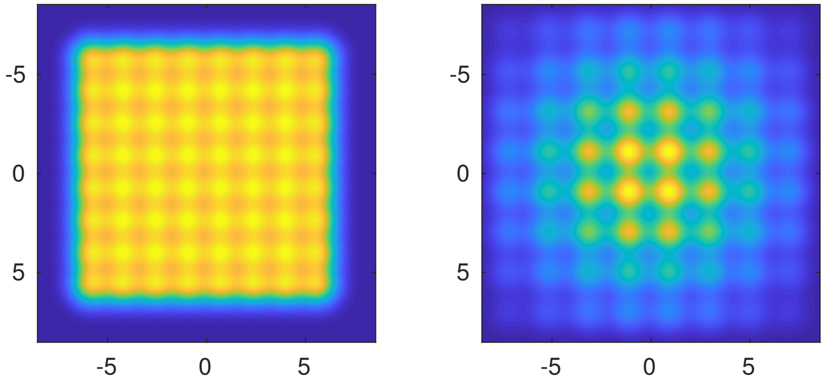

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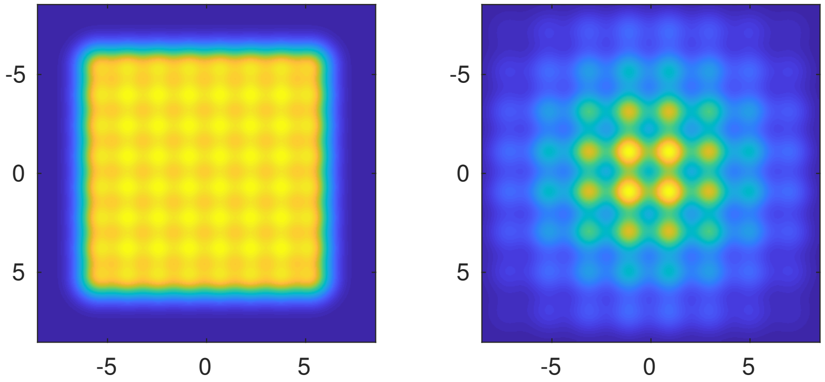

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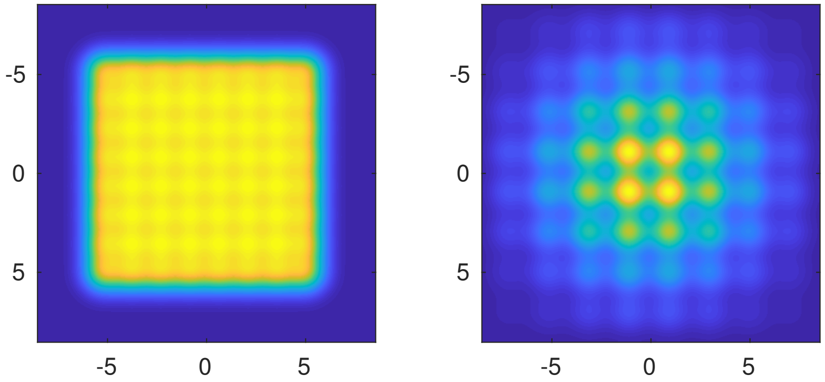

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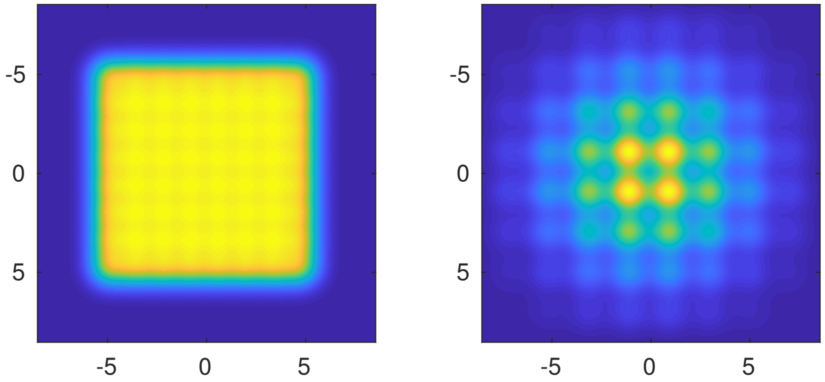

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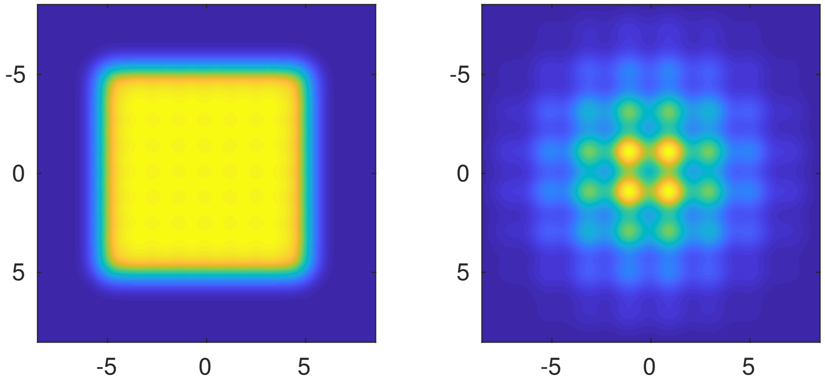

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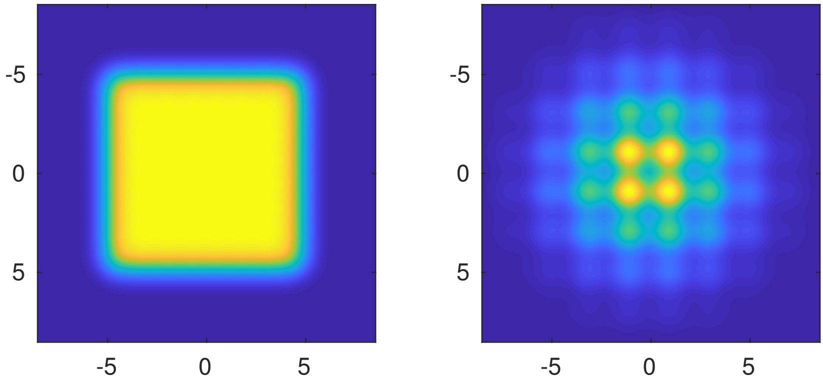
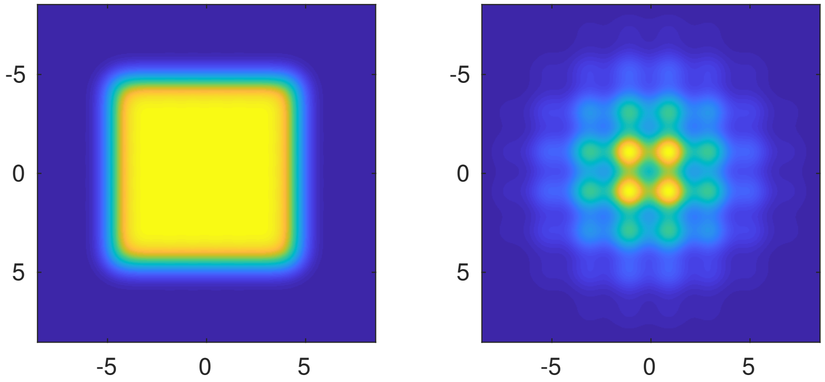


11,5 dB



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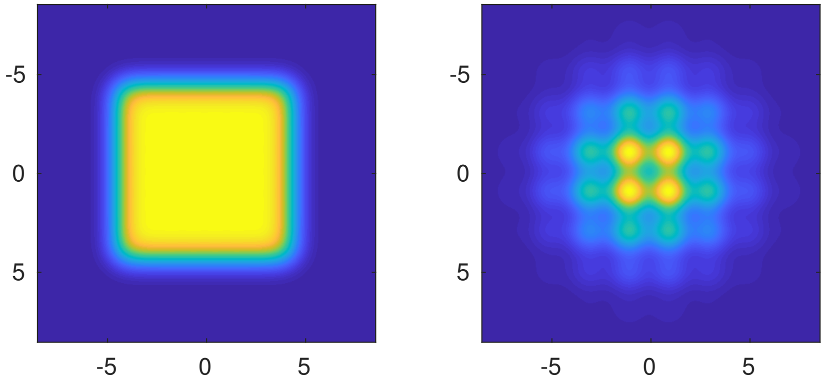
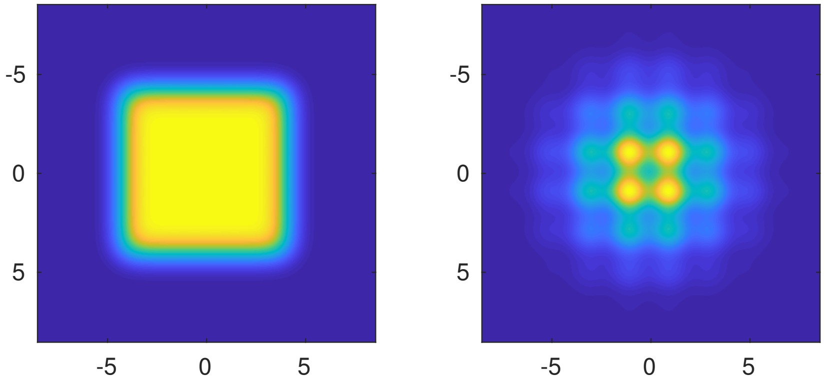
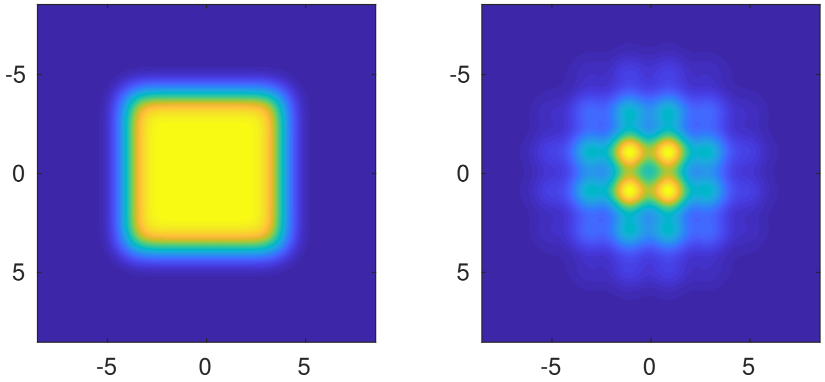


10,5 dB



10,0 dB



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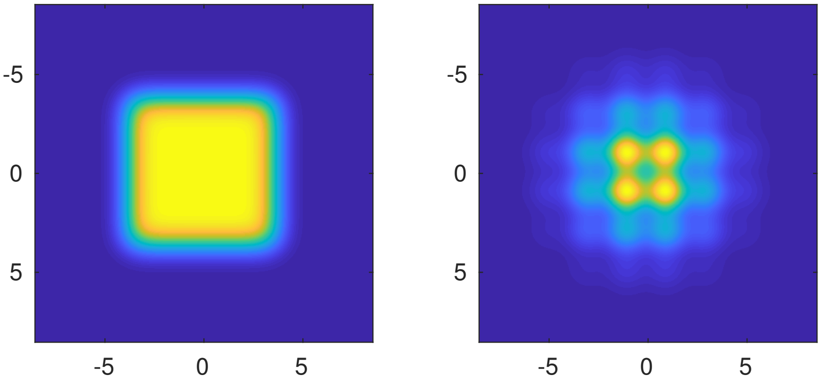

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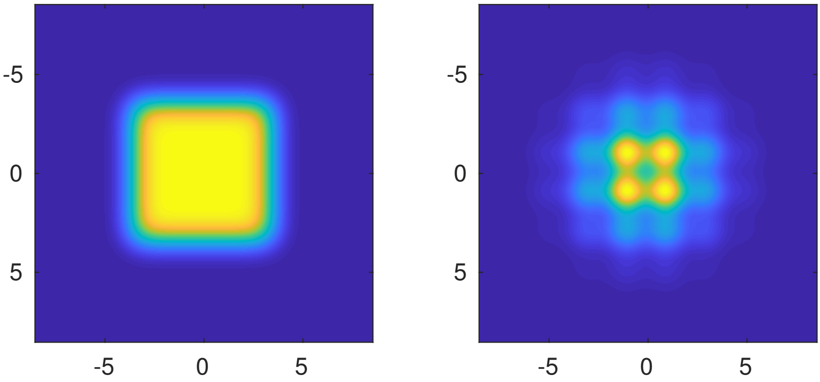

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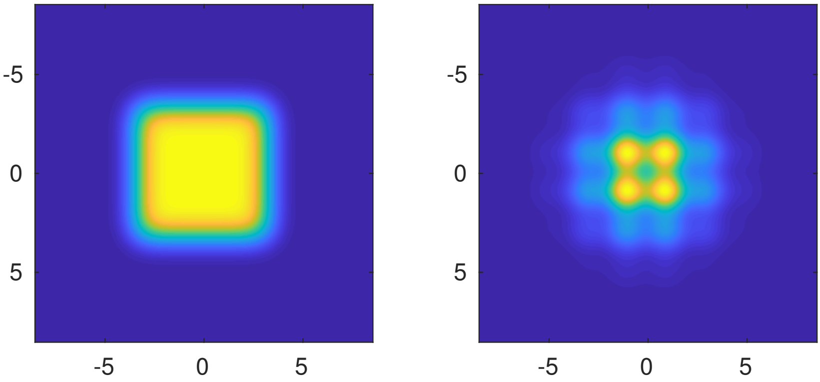

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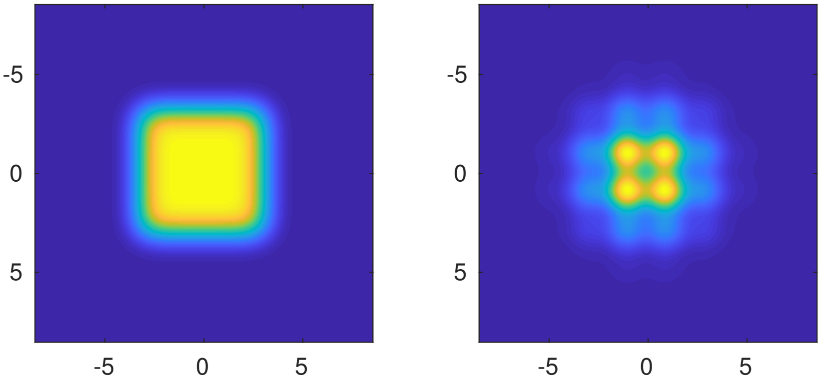

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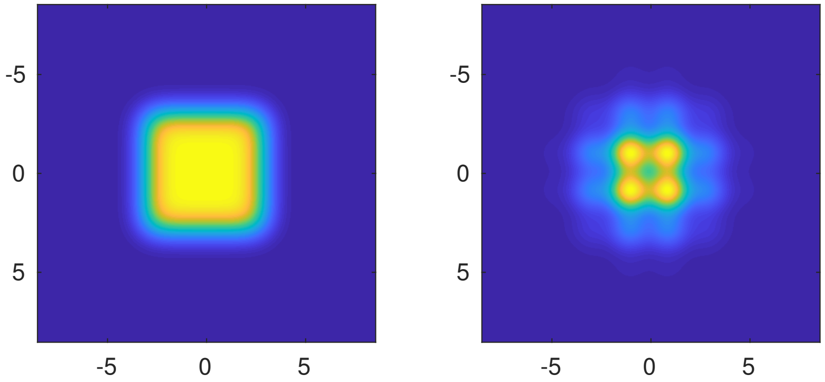

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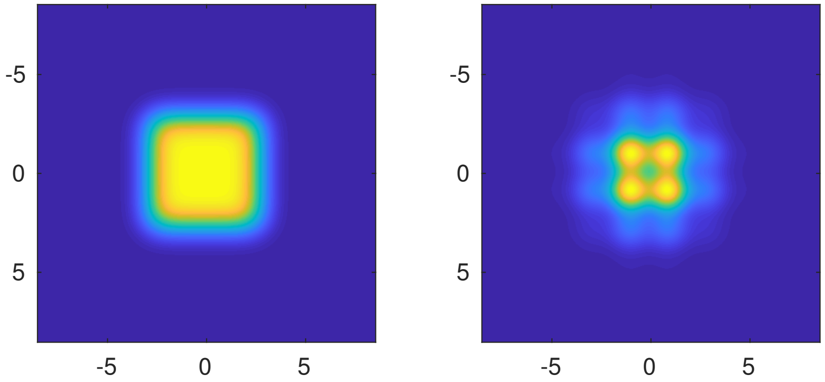
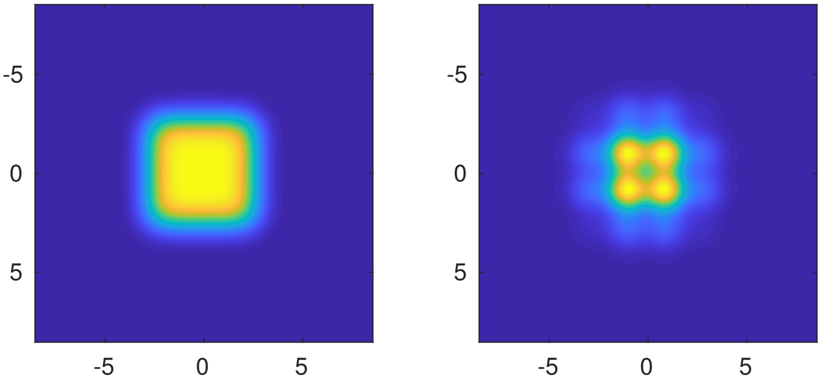
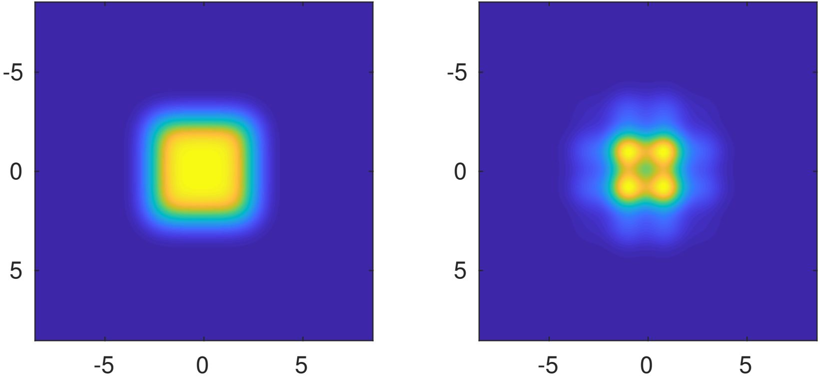


6,5 dB



6,0 dB



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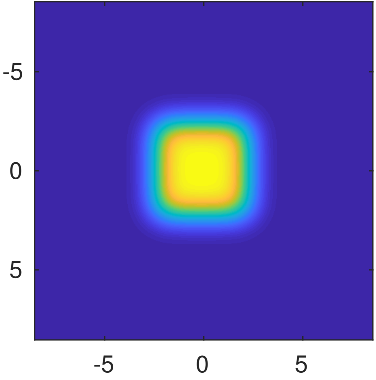

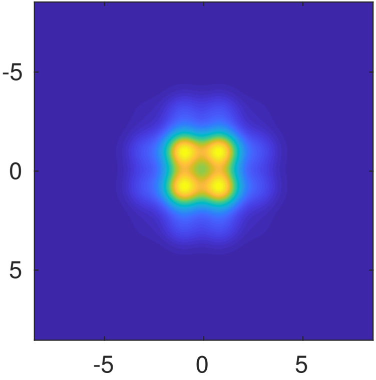

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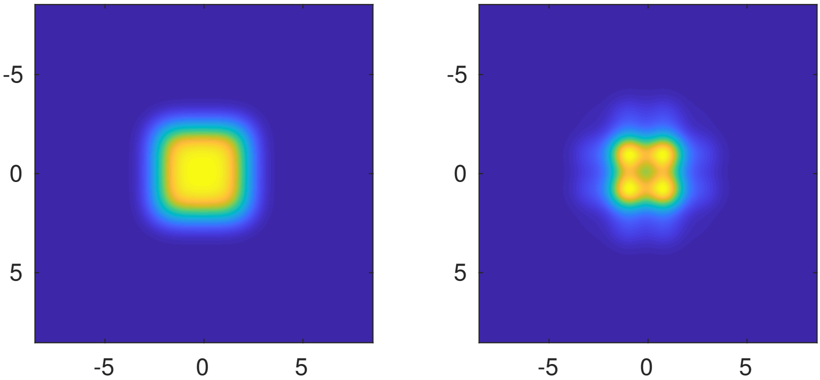
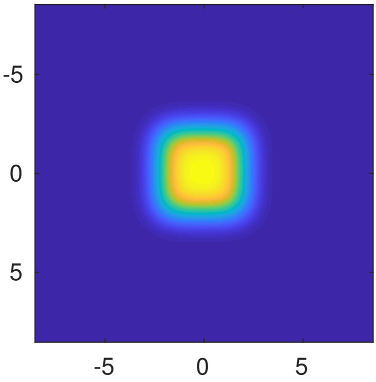
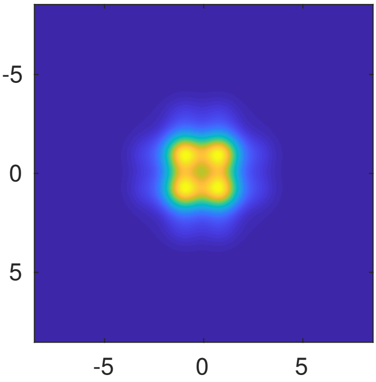


4,5 dB





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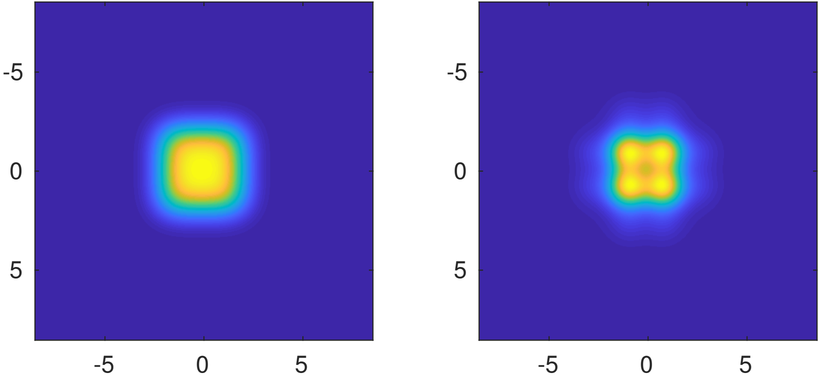

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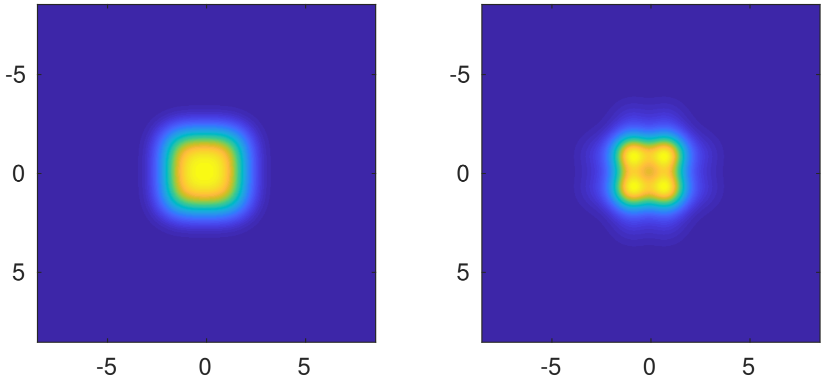

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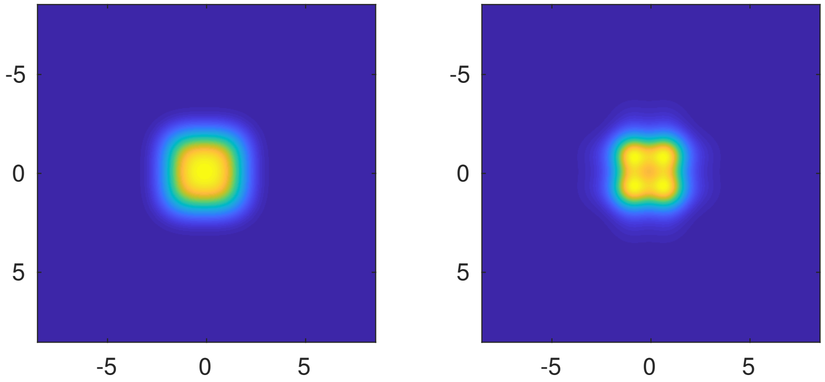
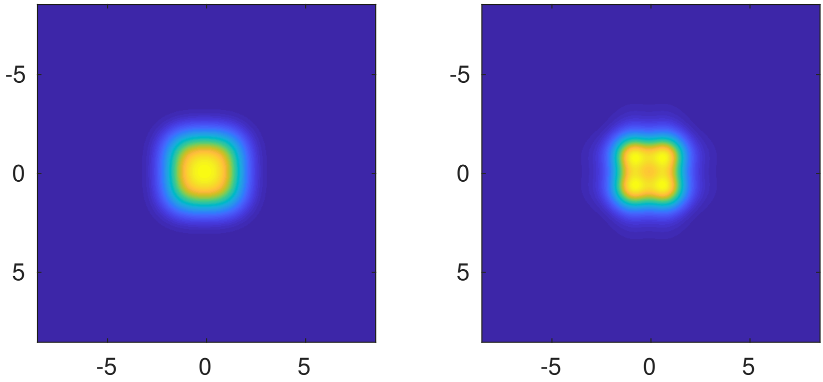


2,5 dB



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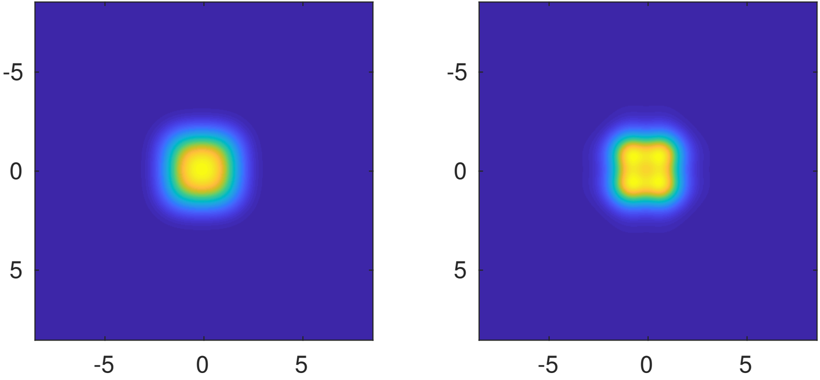
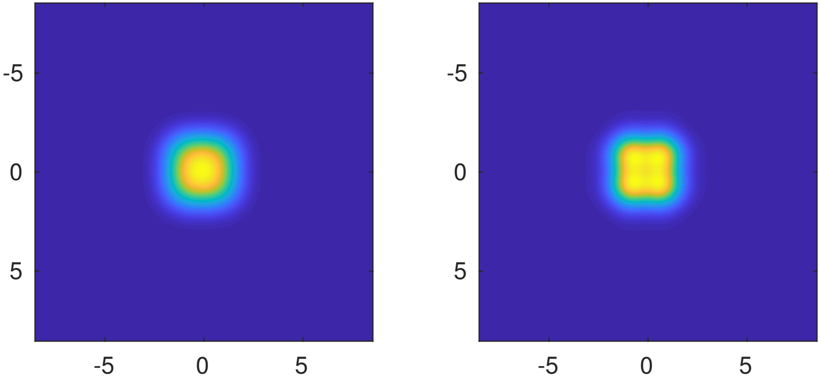


1,5 dB



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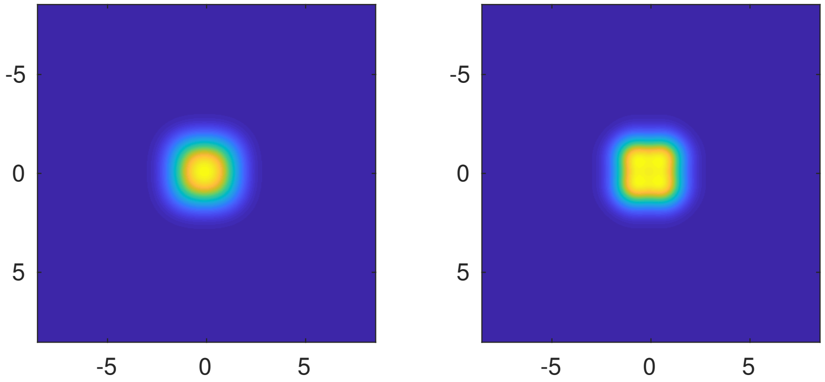

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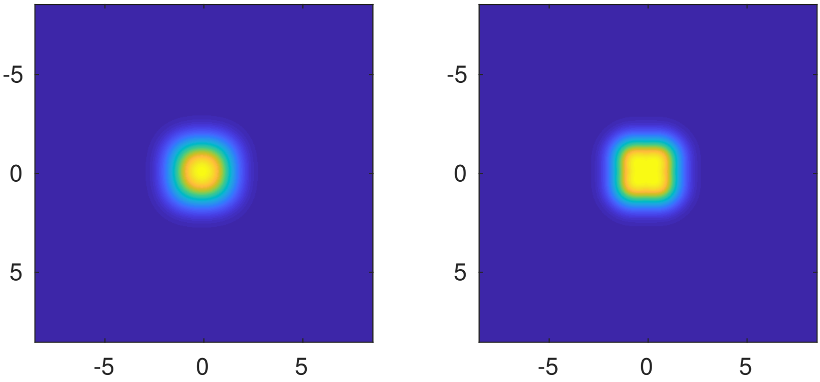
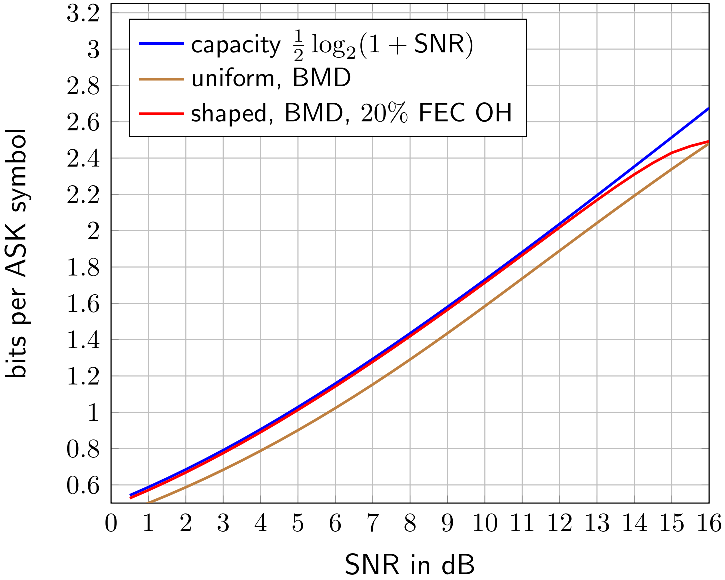
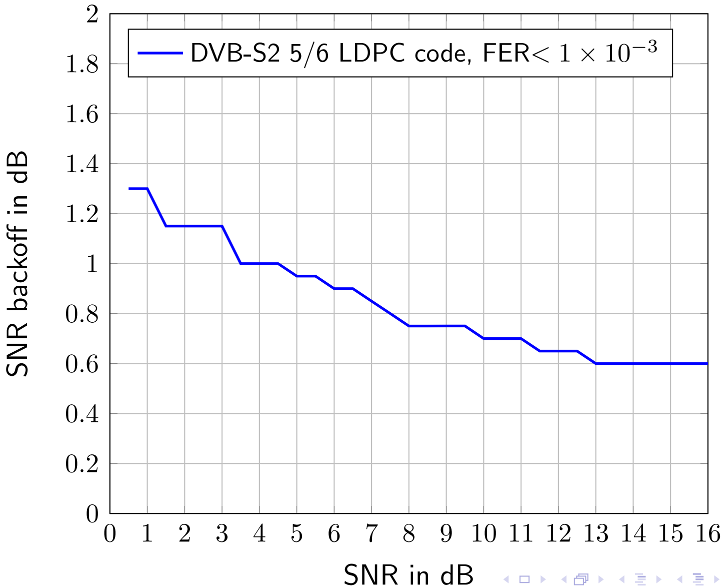


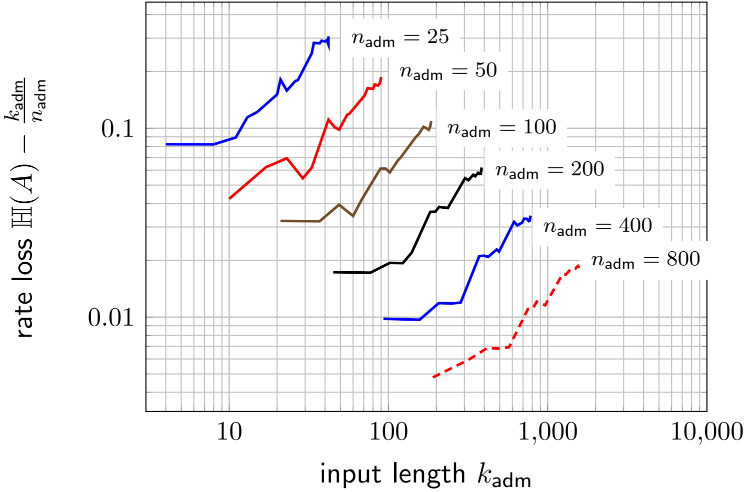
Achievable Rates for PAS with 20% FEC Overhead



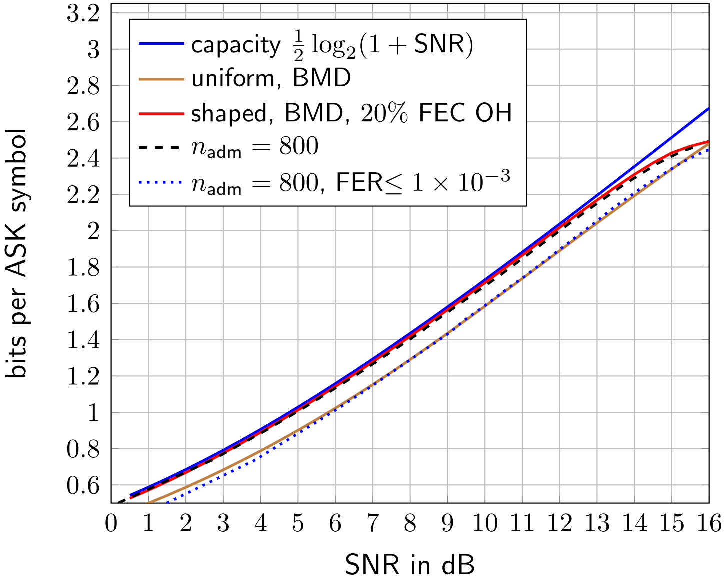
SNR Backoff for 5/6 DVB-S2 LDPC Code at FER$< 1 \times 10^{-3}$



DM Rate Loss


$n_{\text{adm}} = 25$
$n_{\text{adm}} = 50$
$n_{\text{adm}} = 100$
$n_{\text{adm}} = 200$
$n_{\text{adm}} = 400$
$n_{\text{adm}} = 800$
0.1
0.01
10
100
1,000
10,000
rate loss $\mathbb{H}(A) - \frac{k_{\text{adm}}}{n_{\text{adm}}}$
input length k_{adm}

5/6 DVB-S2 LDPC Code, CCDM with $n_{\text{adm}} = 800$



Outline

Conclusions

Presented **Framework** for individually benchmarking

- Decoding metrics: soft/hard/quantized/...
- FEC codes.
- DM algorithms.

Open problem:

- PAS-like architecture for non-coherent transmission.